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XVI.

THEORY OF THE HORIZONTAL PHOTOHELIOGRAPH,

INCLUDING ITS APPLICATION TO THE DETERMINATION OF THE SOLAR PARALLAX BY MEANS OF TRANSITS OF VENUS.

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Presented March 15th, 1877.

The term "Horizontal Photoheliograph" is used to designate that form of Photoheliograph which, it is believed, was first employed by the late Professor Joseph Winlock; and which consists essentially of a fixed telescope whose optical axis is accurately horizontal and in the meridian, and whose objective is directed toward the same side of the zenith as the elevated pole; the sun's rays being reflected into the telescope by a suitable heliostat. The sensitive plate for the reception of the photographic image, is, of course, situated at the chemical focus of the telescope; the plane of the plate being perpendicular to, and its centre coinciding with, the optical axis of tho telescope.

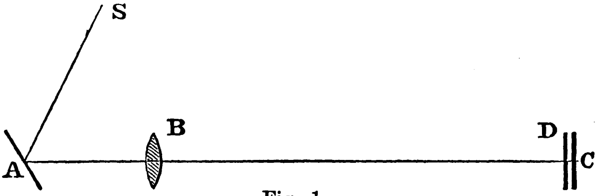

Fig. 1.

The details of the construction of the horizontal photoheliograph, in so far as they are necessary for a proper understanding of the theory of the instrument, are shown in Fig. 1.

A is the heliostat mirror, consisting of a piece of highly polished, but unsilvered glass, whose two surfaces make an angle of about sixty minutes with each other. The front surface is worked as truly plane as possible, and serves to reflect the solar rays through the objective, to the photographic plate. In working the back surface no particular pains are taken, and, on account of its inclination to the front surface, any light reflected from it is thrown entirely away from the photographic plate. B is the objective, which is corrected for the chemical, and not for the visual rays. The distance between the objective and mirror is made as small as possible, consistently with keeping the latter clear of the shadow of the former. D is the reticule, the construction and use of which will be explained presently. C is the photographic plate, the sensitive surface of which faces the objective. The rays from the sun S are reflected by the mirror A through the objective B, and after traversing the reticule D they form an image upon the photographic plate C.

The reticule consists of a system of squares, formed by the intersection of two systems of very fine, straight lines, which are drawn upon one side of, and respectively parallel to the edges of, a thin, square plate of plano-parallel glass; as shown in Fig. 2. In each of these linear systems the number of lines is odd, and the middle line is drawn through the centre of the plate. This reticule is fixed at D, Fig. 1, with its ruled surface toward, parallel to, and two or three millimeters distant from, the sensitive surface of the plate C. Moreover, one of the two systems of lines is set as nearly as possible vertical, and its inclination is accurately determined; and as an additional safeguard, a plumb line, consisting of a silver wire about 0.05 of a millimeter in diameter, is suspended between the reticule and the photographic plate, in such a position that it may hang freely, and at the same time be very nearly in the vertical plane passing through the centres of the reticule and objective. As the light from the objective traverses the reticule before it reaches the photographic plate, the shadow, both of the reticule and of the plumb line, is impressed upon every picture taken with the apparatus; and thus three different ends are gained: Firstly, by comparing the squares of the reticule with the corresponding ones upon the picture, every thing relating to the shrinkage of the collodion can be determined; secondly, the impression of the plumb-line, and also that of the vertical lines, furnishes upon each picture a fixed direction from which to measure angles of position; and, thirdly, the intersection of the middle vertical with the middle horizontal line furnishes a fixed point, which will hereafter be designated as the centre of the

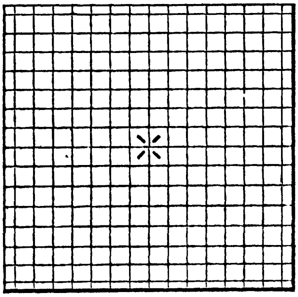

Fig. 2.

plate. To determine the zenith distance and azimuth of this point it is necessary to measure the zenith distance and azimuth of the corresponding point of the reticule, as seen from the second principal point of the objective. For that purpose the mirror A, Fig. 1, is temporarily removed, and a transit instrument is set up in front of the objective B, and in the prolongation of its axis. The objective thus becomes a collimator to the transit instrument, through the eyepiece of which the lines of the reticule may be observed as if they were at an infinite distance. It should be remarked, however, that, as the reticule is slightly inside the visual focus of the objective, it is not generally possible to obtain perfectly distinct vision of its lines and of the wires of the transit at the same time. This difficulty is obviated by marking the intersection of the middle vertical and middle horizontal lines of the reticule in such a distinct manner that the slight mal-adjustment of focus does not prevent it from being seen. Then, by means of the transit, the reticule is adjusted so that the point in question is very approximately in the meridian, and at a zenith distance of ninety degrees. Finally, the exact azimuth and zenith distance of the point are measured.

If the photoheliograph is employed to depict any object whose linear magnitude is such that it subtends an angle a at the first principal point of the objective, and if the centre of the image coincides with the centre of the plate upon which it is taken; then, no matter what may be the solar focal distance of the objective, the magnitude of the image will be

$$2 \tan \tfrac{1}{2} a \, (D + E'' - T) \qquad (1)$$

where D is the distance between the back surface of the objective and the sensitive surface of the photographic plates, E'' the distance from the back surface of the objective to its second principal point; and T a correction due to the thickness of the reticule plate. It is thus evident that, in all calculations involving measurements of pictures taken with this apparatus, the value to be employed as the focal distance of the objective is

$$D + E'' - T \qquad (2)$$

D is obtained by direct measurement, the accuracy of which must be such that the uncertainty of the resulting value will not exceed one part in ten thousand, and it is desirable that it should not exceed one part in forty thousand. To obtain E'' we put

n = refractive index of crown-glass lens.
n' = refractive index of flint-glass lens.

ne = thickness of crown-glass lens.
$n'e'$ = thickness of flint-glass lens.
s = space between the interior surfaces of the crown and flint lenses, measured along the optical axis.
$(n - 1)f'$ = radius of first surface of crown lens. N. B.—This is the surface nearest the heliostat.
$(n - 1)f''$ = radius of second surface of crown lens.
$(n' - 1)f'''$ = radius of first surface of flint lens.
$(n' - 1)f^{\text{IV}}$ = radius of second surface of flint lens.
φ° = focal distance of crown-glass lens.
φ' = focal distance of flint-glass lens.

Then, from the "Dioptrische Untersuchungen" given in "Gauss' Werke," Vol. 5, pp. 262–265, we derive the formulæ

$$\varphi^\circ = \frac{f'f''}{f' + f'' - e} \qquad \varphi' = \frac{f'''f^{\text{IV}}}{f''' + f^{\text{IV}} - e'} \qquad (3)$$

And if the objective is an ordinary double achromatic, corrected either for the visual or chemical rays,

$$\left.\begin{aligned} t' &= s + \frac{e\phi^\circ}{f'} + \frac{e'\phi'}{f^{\text{IV}}} \\ E'' &= \varphi' \left\{ \frac{e'}{f'''} + \frac{t'}{\phi^\circ + \phi' - t'} \right\} \end{aligned}\right\} \qquad (4)$$

But if it is a single lens, then

$$E'' = \frac{e\phi^\circ}{f'} \qquad (5)$$

To find the value of the correction T, let $abcd$, Fig. 3, be a section of the reticule plate, and let ef be a ray of light incident upon it in the plane of the paper. If this ray suffered no refraction, it would emerge from the plate at g; but, owing to refraction, it actually does emerge at h, after which its path is parallel to ef produced. At h erect a perpendicular to bc. It will be intersected at i by ef produced, and the distance hi will be the required value of T. Let t be the thickness of the glass composing the reticule plate; n its index of refraction; and i the angle of incidence of the ray ef. Then

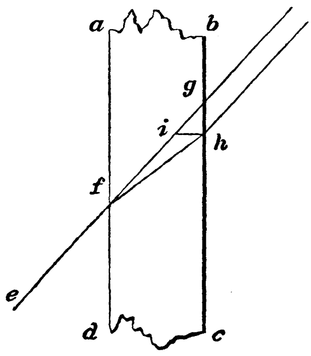

Fig. 3.

$$T = \frac{t(n-1)\cos i}{n} \qquad (6)$$

and as i cannot exceed half a degree, its cosine will be very nearly unity, and it will be sufficiently accurate to write

$$T = \frac{t(n-1)}{n} \qquad (7)$$

If we assume the reticule plate to be of crown glass, and its refractive index to be 1.53, then $T = 0.347\,t$; and it is evident that, in order to make T small, the reticule plate should be as thin as possible.

From the experience thus far had with the horizontal photoheliograph, it appears that, if the focal distance of the objective is F, its clear aperture should be $0.0100\,F$. The clear aperture of the heliostat mirror, which is circular in form, should be $0.0142\,F$. The plates upon which the pictures are taken should be square, and of such a size that their sides, as seen from the centre of the objective, may subtend an angle of about sixty minutes. This should also be the size of the reticule plate, and the distance between any two consecutive lines of the reticule should subtend an angle of about four minutes. Throughout the remainder of this paper it will be assumed that these are the proportions of the apparatus. The actual focal distance of the objectives of the instruments heretofore constructed has generally been about twelve meters.

To avoid repetition, the notation which will be employed throughout the remainder of this paper is here given. Let PA, Fig. 4, be the meridian of the place of observation; P being the pole, and Z the zenith. Let S be the position of the sun as seen from the centre of the earth, and S_* its position as seen from the place of observation. Hereafter, to avoid circumlocution, S will be designated as the true, and S_* as the apparent, sun. Let v be the vertex of the apparent sun, and V the position of Venus as seen from the centre of the earth. Also, let M be the point where the normal to the heliostat mirror pierces the heavens, and V', S', S'_*, and v', the positions of the reflected images of Venus, the true sun, the apparent sun, and the vertex of the apparent sun, as seen from the second principal point of the photographic objective. Then the following notation will be adopted: —

φ = latitude of the place of observation.
φ' = co-latitude of the place of observation = PZ = $90° - \varphi$.
Δ_s = polar distance of true sun = PS.

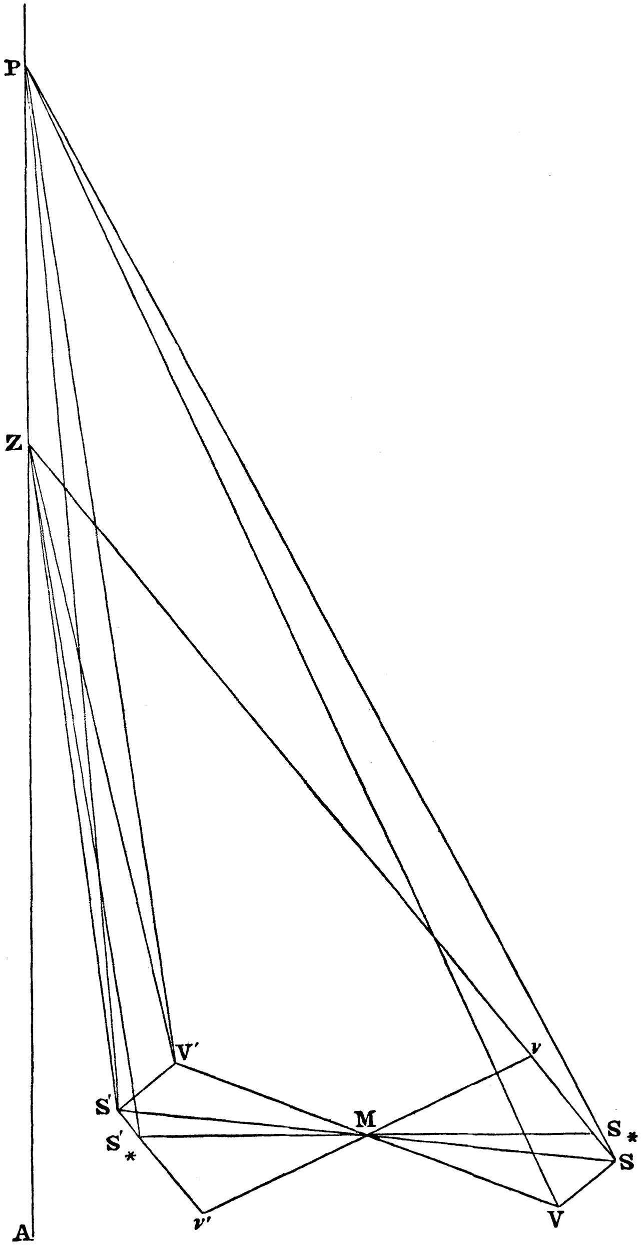

Fig. 4.

Δ_v = polar distance of Venus = PV.
Δ'_s = polar distance of the reflected image of the true sun = PS'.
Δ'_v = polar distance of the reflected image of Venus = PV'.
t_s = hour angle of true sun = APS.
t_v = hour angle of Venus = APV.
t'_s = hour angle of the reflected image of the true sun = APS'.
t'_v = hour angle of the reflected image of Venus = APV'.
ζ_s = zenith distance of true sun = ZS.
ζ'_s = zenith distance of the reflected image of the true sun = ZS'.
ζ''_s = zenith distance of the reflected image of the apparent sun = ZS'_*.
ζ'_v = zenith distance of the reflected image of Venus = ZV'.
A_s = azimuth of the true sun = AZS.
A'_s = azimuth of the reflected image of the true sun = AZS'.
A''_s = azimuth of the reflected image of the apparent sun = AZS'_*.
A'_v = azimuth of the reflected image of Venus = AZV'.
θ = the angle ZS'_*S'.
r = the refraction.
π' = the parallax in altitude. Hence $r - \pi' = SS_* = S'S'_*$.
β = the angle $ZS'V'$.
γ = the angle $ZV'S'$.
λ = the angle $PS'V'$.
ϱ = the geocentric distance from centre of sun to centre of Venus = $S\,V'$ = SV.
ω = the geocentric position angle of Venus relatively to the sun's centre = PSV.
σ = the angle PVS.
χ = the angle PSS'.
ψ = the angle $PS'S$.

We have next to show how pictures obtained with the horizontal photoheliograph may be measured, and the results expressed in terms of some one of the systems of spherical co-ordinates usually employed in astronomy.

Upon an engine arranged to give polar co-ordinates, the plate to be measured is carefully adjusted, so that the centre of the image of the sun coincides as accurately as possible with the origin of co-ordinates. The fixed line, from which the angles of the co-ordinates are to be measured, is taken perpendicular to the image of the plumb line; the zero being toward that edge of the plate which was on the right hand, as viewed from the object glass, when the picture was taken, and (the collodion side of the plate being nearest the observer) a motion of the radius vector in the opposite direction to that of the hands of a watch corresponding to increasing angles. Then the distance from the origin of co-ordinates to the sun's limb is measured at every thirty degrees, throughout the whole circumference; and the distances and angles to the centre of Venus, and to the centre of the plate, are also measured.

Owing to the effect of refraction, the outline of the sun's image will not be circular, but may, with sufficient accuracy, be assumed to be an ellipse whose minor axis coincides with the vertical circle passing through the sun's centre. To find the position of the image of this vertical circle upon the photographic plate, we refer to Fig. 4; and remark that, owing to the equality of the angles of incidence and reflection, $S_{*}M = S'_{*}M$, $vM = v'M$, and the angle $S_{*}Mv$ is equal to the angle $S'_{*}Mv'$. Hence $vS_{*}S'_{*} = S_{*}S'_{*}v'$, and $ZS'_{*}v' = ZS'_{*}S_{*} + S_{*}S'_{*}v' = ZS'_{*}S_{*} + ZS_{*}S'_{*}$. But ZS'_{*} is the direction of the plumb line upon the plate, and $S'_{*}v'$ is the semi-minor axis of the elliptical image of the sun. To find the value of the angle $ZS'_{*}v'$ we require the zenith distance and azimuth of the true sun, which are given by the formulæ (8); the necessary data being, the exact instant at which the picture was taken, the right ascension and polar distance of the sun, and the latitude of the place of observation.

$$\left.\begin{aligned}\tan M &= \frac{\cot \Delta_s}{\cos t_s}\\ \tan A_s &= \frac{\tan t_s \cos M}{\sin(\phi - M)}\\ \tan \zeta_s &= \frac{\tan(\phi - M)}{\cos A_s}\end{aligned}\right\} \qquad (8)$$

The zenith distance, ζ''_s, and azimuth A''_s, of the reflected image of the apparent sun, are found by methods explained in a subsequent part of this paper. Then, in the spherical triangle $ZS_{*}S'_{*}$, the angle $S_{*}ZS'_{*}$ is equal to $A_s \sim A''_s$, $ZS'_{*} = \zeta''_s$, and $ZS_{*} = \zeta_s - r + \pi'$, where r and π' are the refraction, and parallax in altitude. Whence, putting the angle $ZS'_{*}v' = 180° - \theta$,

$$\cot \tfrac{1}{2}\theta = \frac{\cos \frac{1}{2}(\zeta''_s - \zeta_s + r - \pi')}{\cos \frac{1}{2}(\zeta''_s + \zeta_s - r + \pi')} \cot \tfrac{1}{2}(A_s \sim A''_s) \qquad (9)$$

Thus the angle upon the photographic plate, between the fixed line of reference and the major axis of the image of the apparent sun, will be θ; and if R and ε are the polar co-ordinates of any point of the limb of this image, its rectangular co-ordinates, referred to its axes and approximate centre will be

$$\left.\begin{array}{l} y' = R \sin(\varepsilon \mp \theta) \\ x' = R \cos(\varepsilon \mp \theta) \end{array}\right\} \qquad (10)$$

in which the upper signs are to be taken when the sun is west, and the lower when it is east, of the meridian.

Owing to atmospheric disturbances, the image of the sun's limb will generally be more or less irregular, and no matter how carefully it may be centred on the measuring engine, the point from which the measures are made will seldom coincide quite accurately with the centre of the image. On this account, to obtain the co-ordinates of any point of the limb referred rigorously to the centre and axes, the equations (10) must be corrected by the small quantities δy and δx, and thus we get

$$\left.\begin{array}{l} y = R \sin(\varepsilon \mp \theta) + \delta y \\ x = R \cos(\varepsilon \mp \theta) + \delta x \end{array}\right\} \qquad (11)$$

The image of the sun's limb being taken as an ellipse, its equation will be

$$0 = - A^2B^2 + A^2y^2 + B^2x^2 \qquad (12)$$

where A and B are respectively the horizontal and vertical semi-diameters of the sun, affected by refraction and parallax. Putting $B^2 = nA^2$, substituting this value in equation (12), and dividing by A^2, we get

$$0 = - nA^2 + y^2 + nx^2 \qquad (13)$$

Finally, substituting the values of y and x from the equations (11), neglecting the squares of δy and δx, and reducing, we obtain

$$0 = \left\{\begin{array}{l} + \frac{1}{2}[\sin^2(\varepsilon \mp \theta) + n \cos^2(\varepsilon \mp \theta)]\, R \\ - \frac{n}{2R} A^2 + \sin(\varepsilon \mp \theta)\, \delta y \\ + n \cos(\varepsilon \mp \theta)\, \delta x \end{array}\right\} \qquad (14)$$

Each measurement of the sun's limb, made upon the photograph, gives one equation of the form (14), and from all the equations thus obtained the values of A, δy, and δx are found by the method of least squares.

The contraction of the sun's vertical semi-diameter on account of refraction will be $\frac{1}{2}(r' - r')$; where r' and r'' are respectively the refraction of the lower and upper limbs. The contraction of the horizontal semi-diameter on account of refraction, for all zenith distances less than 85°, may be taken as constant and equal to $0''.25$. Hence, if s represents the sun's semi-diameter, we have

$$n = \left(\frac{s - \frac{1}{2}(r' - r'')}{s - 0''.25}\right)^2 \qquad (15)$$

which is the value to be employed in equation (14).

As changes of refraction are not strictly proportional to changes of zenith distance, the centre of the sun's image will not coincide rigorously with the image of the sun's centre. Let the distance between those two points be δr, and let r''' be the refraction of the sun's centre, r'' and r' being respectively the refraction of his upper and lower limbs, as before. Then

$$\delta r = \tfrac{1}{2}(r' + r'') - r''' \qquad (16)$$

and the co-ordinates of the image of the sun's centre are $\delta y + \delta r$, and δx. These rectangular co-ordinates are transformed into polar co-ordinates of our original system by means of the formulæ

$$\left.\begin{aligned} R &= [(\delta y + \delta r)^2 + (\delta x)^2]^{\frac{1}{2}} \\ \sin \eta &= \frac{\delta y + \delta r}{R} \\ \cos \eta &= \frac{\delta x}{R} \\ \varepsilon &= \eta \pm \theta \end{aligned}\right\} \qquad (17)$$

where the double sign is to be taken in the same way as in equations (10).

The polar co-ordinates of the image of the centre of the apparent sun have thus been found; and our original measurements gave the polar co-ordinates of the centres of the image of the apparent Venus and of the photographic plate. Let R, R', R'' be, respectively, the radii vectores, and ε, ε', ε'' the angles of these co-ordinates. Passing now to a system of rectangular co-ordinates whose origin is at the centre of the plate, and whose axis of X is parallel to the fixed line of the original system of polar co-ordinates; we get for the image of the centre of the apparent sun,

$$\left.\begin{aligned} y &= R \sin \varepsilon - R'' \sin \varepsilon'' \\ x &= R \cos \varepsilon - R'' \cos \varepsilon'' \end{aligned}\right\} \tag{18}$$

and for the image of the centre of the apparent Venus

$$\left.\begin{aligned} y' &= R' \sin \varepsilon' - R'' \sin \varepsilon'' \\ x' &= R' \cos \varepsilon' - R'' \cos \varepsilon'' \end{aligned}\right\} \tag{19}$$

Let it be assumed that when the picture was taken ζ_0 and A_0 were the zenith distance and azimuth of the centre of the plate, as seen from the second principal point of the objective; and that F was the reduced distance between the latter point and the sensitive surface of the plate, or in other words,

$$F = D + E'' - T \tag{20}$$

as given by equation (2). Further, let the angles subtended at the second principal point of the objective by the co-ordinates, y, x, y', x', of the equations (18) and (19) be, respectively, $\delta\zeta_s$, δA_s, $\delta\zeta_v$, δA_v. Then, as the plate was perpendicular to, and its centre coincided with, the optical axis of the objective,

$$\left.\begin{aligned} \tan \delta\zeta_s &= \frac{R \sin \epsilon - R'' \sin \epsilon''}{F} \\ \tan \delta A_s &= \frac{R \cos \epsilon - R'' \cos \epsilon''}{F} \\ \tan \delta\zeta_v &= \frac{R' \sin \epsilon' - R'' \sin \epsilon''}{F} \\ \tan \delta A_v &= \frac{R' \cos \epsilon' - R'' \cos \epsilon''}{F} \end{aligned}\right\} \tag{21}$$

Denoting by ζ''_s, A''_s, ζ''_v, A''_v, the apparent zenith distances and azimuths of the reflected images of the sun and Venus, as seen from the second principal point of the objective, we now have

$$\left.\begin{aligned} \zeta''_s &= \zeta_0 - \delta\zeta_s \\ A''_s &= A_0 + \delta A_s \\ \zeta'_v &= \zeta_0 - \delta\zeta_v \\ A''_v &= A_0 + \delta A_v \end{aligned}\right\}. \tag{22}$$

The next step will be to free these zenith distances and azimuths from the effects of refraction and parallax. For that purpose, consider the spherical triangle $ZS'S'_*$, Fig. 4; but instead of limiting the points S, S_*, S', S'_*, to the sun, let them represent any heavenly body whatever. Then, $ZS' = \zeta'$; $ZS'_* = \zeta''$; $S'S'_* = SS_* = r - \pi'$; $S'ZS'_* = A'' \sim A'$; and $ZS'_*S' = \theta$, the value of which is given by equation (9). The relations subsisting among these parts are,

$$\left.\begin{aligned} \tan m &= \tan (r - \pi') \cos \theta \\ \sin \zeta' \cos (A'' \sim A') &= \frac{\cos (r - \pi') \sin (\zeta'' - m)}{\cos m} \\ \sin \zeta' \sin (A'' \sim A') &= \sin (r - \pi') \sin \theta \end{aligned}\right\} \qquad (23)$$

To simplify these equations we remark that $(r - \pi')$ will rarely amount to $5'$, and as $\cos \theta$ must always be less than unity, we may write with all needful accuracy

$$m = (r - \pi') \cos \theta \qquad (24)$$

ζ' will never differ from $90°$ by so much as $30'$, and therefore its sine may be taken as unity; while as $(A'' \sim A')$ can scarcely amount to $5'$, and will usually be far less, we may write unity for its cosine, and substitute the arc for its sine. We thus find

$$\left.\begin{aligned} \zeta' &= \zeta'' - (r - \pi') \cos \theta \\ A' &= A'' \mp (r - \pi') \sin \theta \end{aligned}\right\} \qquad (25)$$

As these equations are perfectly general, we have only to substitute in them, for ζ'' and A'', the apparent zenith distances and azimuths of the images of the sun and Venus, given by the equations (22), and there results the true zenith distances and azimuths of the images of these bodies, which are

$$\left.\begin{aligned} \zeta'_s &= \zeta_0 - \delta\zeta_s - (r_s - \pi'_s) \cos \theta_s \\ A'_s &= A_0 + \delta A_s \mp (r_s - \pi'_s) \sin \theta_s \\ \zeta'_v &= \zeta_0 - \delta\zeta_v - (r_v - \pi'_v) \cos \theta_v \\ A'_v &= A_0 + \delta A_v \mp (r_v - \pi'_v) \sin \theta_v \end{aligned}\right\} \qquad (26)$$

in which the upper signs are to be taken when the body is west, and the lower when it is east, of the meridian. Strictly speaking, the value of θ will not be the same for Venus as for the sun, but the difference will generally be so slight that either value may be employed for both bodies.

The effect of parallax in azimuth in displacing Venus relatively to the sun's centre can never exceed $0''.08$, and will usually be much less; but, if it is thought desirable to correct the equations (26) on that account, the mode of doing so will be obvious when it is remembered that it will be sufficiently accurate to consider the parallax in azimuth as acting perpendicularly to the parallax in altitude.

From the zenith distances and azimuths given by the equations (26), the corresponding polar distances and hour angles must next be found. The rigorous formulæ for this purpose are

$$\left.\begin{aligned} \tan m &= \tan \zeta' \cos A' \\ \tan t &= \frac{\tan A' \sin m}{\cos(\phi - m)} \\ \operatorname{cotan} \Delta &= \tan(\varphi - m) \cos t \end{aligned}\right\} \qquad (27)$$

but A' will generally be so small that its cosine may be taken as unity, and then we may write

$$\left.\begin{aligned} \tan t &= \frac{\tan A' \sin \zeta'}{\cos(\phi - \zeta')} \\ \operatorname{cotan} \Delta &= \tan(\varphi - \zeta') \cos t \end{aligned}\right\} \qquad (28)$$

In the spherical triangle PS′V′, Fig. 4, we have the relations

$$\left.\begin{aligned} \tan m &= \tan \Delta'_v \cos(t'_s \sim t'_v) \\ \sin \varrho \cos \lambda &= \frac{\cos \Delta'_v}{\cos m} \sin(\Delta'_s - m) \\ \sin \varrho \sin \lambda &= \sin \Delta'_v \sin(t'_s \sim t'_v) \end{aligned}\right\} \qquad (29)$$

Usually it will be sufficiently accurate to put $\cos(t'_s \sim t'_v) = 1$, and then $m = \Delta'_v$, and these equations become

$$\left.\begin{aligned} \sin \varrho \cos \lambda &= \sin(\Delta'_s - \Delta'_v) \\ \sin \varrho \sin \lambda &= \sin \Delta'_v \sin(t'_s \sim t'_v) \end{aligned}\right\} \qquad (30)$$

from which ϱ and λ are obtained. As a check, ϱ may be computed directly from the zenith distances and azimuths furnished by the equations (26), the requisite formulæ being

$$\left.\begin{aligned} \tan B &= \frac{(A'_s \sim A'_v)\sin\zeta'_s}{\zeta'_v - \zeta'_s} \\ \varrho &= \frac{(A'_s \sim A'_v)\sin\zeta'_s}{\sin B} = \frac{\zeta'_v - \zeta'_s}{\cos B} \end{aligned}\right\} \qquad (31)$$

From the spherical triangle PSS′, Fig. 4, we get

$$\left.\begin{aligned} \tan\tfrac{1}{2}(\chi + \psi) &= \frac{\cos\frac{1}{2}(\Delta'_s - \Delta_s)}{\cos\frac{1}{2}(\Delta'_s + \Delta_s)}\cot\tfrac{1}{2}(t'_s \sim t_s) \\ \tan\tfrac{1}{2}(\chi - \psi) &= \frac{\sin\frac{1}{2}(\Delta'_s - \Delta_s)}{\sin\frac{1}{2}(\Delta'_s + \Delta_s)}\cot\tfrac{1}{2}(t'_s \sim t_s) \end{aligned}\right\} \qquad (32)$$

Referring again to Fig. 4, it is evident that SV is the distance, and PSV the position angle of Venus from the centre of the sun. To find the values of these quantities we have

$$\left.\begin{aligned} \mathrm{SV} &= \mathrm{S'V'} = \varrho \\ \mathrm{PSV} &= \mathrm{PSS'} \pm \mathrm{S'SV} = \mathrm{PSS'} \pm \mathrm{SS'V} = \mathrm{PSS'} \pm \mathrm{PS'S} \mp \mathrm{PS'V'} \end{aligned}\right\} \qquad (33)$$

But PSV $= \omega$, PSS′ $= \chi$, PS′S $= \psi$, PS′V′ $= \lambda$, and thus we get

$$\omega = (\chi \pm \psi) \mp \lambda \qquad (34)$$

in which the upper signs are to be taken when PS′S is greater than PS′V′, and the lower when PS′S is less than PS′V′. If it is assumed that the pole which forms part of the triangle PSV is always the elevated one; and also that position angles are counted from the north around by the east; then, in the northern hemisphere, when the sun is east of the meridian the position angle will be $360° - \omega$, while west of the meridian it will be ω; and in the southern hemisphere, when the sun is east of the meridian the position angle will be $180° + \omega$, while west of the meridian it will be $180° - \omega$.

Finally, if the polar distance of Venus, and the difference between her right ascension and that of the sun are required, these quantities may be obtained from the spherical triangle PSV by means of the formulæ

$$\left.\begin{aligned} \tan n &= \tan\varrho\cos\omega \\ \tan\Delta_v\cos(\alpha_s \sim \alpha_v) &= \tan(\Delta_s - n) \\ \tan\Delta_v\sin(\alpha_s \sim \alpha_v) &= \frac{\sin n\tan\omega}{\cos(\Delta_s - n)} \end{aligned}\right\} \qquad (35)$$

in which α_s and α_v are respectively the right ascensions of the sun and of Venus.

In the preceding development of the theory of the horizontal photoheliograph, continual reference has been made to the centres of the sun and Venus, but of course it will be understood that all the equations apply equally well to any other pair of celestial objects which may have been photographed with the same apparatus.

As the horizontal photoheliograph was much used in observing the last transit of Venus, it is perhaps desirable to give here a direct method of deducing the solar parallax from the photographs then obtained. For that purpose consider the quadrilateral PZS′S, which is composed of the triangles PZS′ and PS′S. In the triangle PZS′ we have the relation

$$\sin A'_s \cot B = \sin \zeta'_s \cot \varphi' + \cos \zeta'_s \cos A'_s \qquad (36)$$

in which φ' is the colatitude, ZP; and B is the angle ZS′P. Considering all the parts, except φ', as variable, and differentiating, we get

$$\left.\begin{aligned} dB = &- \frac{\sin^2 B}{\sin A'_s} \cos \zeta'_s \left(\cot \varphi' \, d\zeta'_s + \sin A'_s \, dA'_s\right) \\ &+ \cos A'_s \left(\frac{\sin^2 B}{\sin A'_s} \sin \zeta'_s \, d\zeta'_s - \frac{\sin B}{\sin A'_s} \cos B \, dA'_s\right) \end{aligned}\right\} \qquad (37)$$

To obtain approximately the maximum value of this differential, we remark that A'_s can never exceed ± 13′; and as sin B must always be less than sin A'_s, $\frac{\sin B}{\sin A'_s}$ must be less than unity; and $\frac{\sin^2 B}{\sin A'_s}$ must be less than sin A'_s, that is, it must be less than 0.004. ζ'_s can never differ from 90° by more than ± 13′, and therefore its sine may be taken as unity, and its cosine cannot exceed 0.004. If the latitude of the place of observation is less than 50°, φ' will be greater than 40°, and its cotangent will be less than 1.20. Substituting these values in the second member of equation (37), all the terms except the last become evanescent, and we may write, without an error of one part in ten thousand,

$$dB = - dA'_s \qquad (38)$$

But the only way in which A'_s can be made to vary is by varying the adopted value of the solar parallax, π_s. Hence, as ZS′ is nearly 90°, dA'_s is the resolved value of $d\pi_s$, and as it can never exceed that quantity, it is safe to write

$$dB = - d\pi_s \qquad (39)$$

In the triangle PSS′ we have the relations

$$\left.\begin{aligned}\cos\chi &= -\cos\psi\cos(t_s \sim t'_s) + \sin\psi\sin(t_s \sim t'_s)\cos\Delta_s\\ \cos\psi &= -\cos\chi\cos(t_s \sim t'_s) + \sin\chi\sin(t_s \sim t'_s)\cos\Delta'_s\end{aligned}\right\}\quad(40)$$

Considering all the parts as variable, differentiating, and reducing, we get

$$\left.\begin{aligned}d\chi &= \sin\Delta'_s\sin(t_s \sim t'_s)\,d\Delta_s - \cos\Delta'_s\,d(t_s \sim t'_s) - \cos \mathrm{SS}'\,d\psi\\ d\psi &= \sin\Delta_s\sin(t_s \sim t'_s)\,d\Delta'_s - \cos\Delta_s\,d(t_s \sim t'_s) - \cos \mathrm{SS}'\,d\chi\end{aligned}\right\}\quad(41)$$

Adding, this becomes

$$d\chi + d\psi = \frac{\sin(t_s \sim t'_s)(\sin\Delta_s d\Delta'_s + \sin\Delta'_s d\Delta_s) - d(t_s \sim t'_s)(\cos\Delta_s + \cos\Delta'_s)}{1 + \cos \mathrm{SS}'}\quad(42)$$

To obtain approximately the maximum value of this differential, we remark that at the time of the transit the sun's north polar distance was 112° 49′, and therefore $\sin\Delta_s = 0.922$, and $\cos\Delta_s = 0.388$. If the latitude of the place of observation is not greater than 50°, the value of Δ'_s will lie between 130° and 180°; and consequently its sine will not exceed 0.766, and its cosine cannot be greater than unity. Sin $(t_s \sim t'_s)$ cannot exceed unity. Further, as the triangle PSS′ can only be varied by varying the assumed value of the solar parallax,* $d\Delta_s$ and $d\Delta'_s$ are the resolved values, and $d\ (t_s \sim t'_s)$ is the sum of two resolved values of $d\pi_s$. It is therefore certain that $d\Delta_s$ and $d\Delta'_s$ are not greater than $d\pi_s$, and that $d(t_s \sim t'_s)$ is not greater than $2\,d\pi_s$. Substituting these values in equation (42), and adding all the terms, without regard to sign, we get

$$d\chi + d\psi = \frac{4\frac{1}{2}\,d\pi_s}{1 + \cos \mathrm{SS}'}\quad(43)$$

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* Strictly speaking, although the point S′ can only be varied by varying the assumed value of the solar parallax, the point S can be varied, not only in that way, but also by varying the tabular place of the sun. In practice it will probably be best to neglect at first the errors of the solar tables, and afterwards, when they become known from the solution of the final equations, to compute rigorously the value of $d\chi + d\psi$ for each photograph, by means of (42), and in all cases where it exceeds two or three seconds, which will rarely happen, the corresponding conditional equations of the form of (55) and (56) may be corrected so as to accord with the new values of the solar elements, and a second solution will give very accurate results.

Hence if SS′ does not exceed 120° the value of $d\chi + d\psi$ cannot be so great as $9\,d\pi_s$. Adding to this the value of dB, from equation (39), we find that, under the circumstances specified above, the total variation of the sum of the angles ZS′S and PSS′ cannot be so great as $10\,d\pi_s$. But it is not possible that the value of the solar parallax now generally adopted can be in error by so much as $0''.2$, and therefore the value of $10\,d\pi_s$ cannot be so great as $2''$ and will probably be less than $1''$.

Referring to the figure, it is evident that

$$\mathrm{ZS'V'} + \mathrm{PSV} = \mathrm{ZS'S} + \mathrm{PSS'} \quad \text{or}$$

$$\beta + \omega = \mathrm{ZS'S} + \mathrm{PSS'} \qquad (44)$$

As the angle β must be obtained from measurements made upon a photograph, it is not probable that it can be depended upon to within $5''$. It has just been shown that the right-hand member of (44) will not be vitiated so much as $2''$ by any possible error in the adopted value of the solar parallax. It therefore follows that the left-hand member of (44) may be regarded as constant, within the limits of error of observation, and thus it appears that

$$-\,d\beta = d\omega \qquad (45)$$

In the triangle ZS′V′ we have

$$\left.\begin{aligned} \tan \tfrac{1}{2}(\beta + \gamma) &= \frac{\cos \frac{1}{2}(\zeta'_\nu - \zeta'_s)}{\cos \frac{1}{2}(\zeta'_\nu + \zeta'_s)} \cot \tfrac{1}{2}(A'_s \sim A'_\nu) \\ \tan \tfrac{1}{2}(\beta - \gamma) &= \frac{\sin \frac{1}{2}(\zeta'_\nu - \zeta'_s)}{\sin \frac{1}{2}(\zeta'_\nu + \zeta'_s)} \cot \tfrac{1}{2}(A'_s \sim A'_\nu) \end{aligned}\right\} \qquad (46)$$

from which β and γ are derived. In the same triangle we also have the relations

$$\left.\begin{aligned} \cos \varrho &= \cos \zeta'_\nu \cos \zeta'_s + \sin \zeta'_\nu \sin \zeta'_s \cos (A'_s \sim A'_\nu) \\ \sin (A'_s \sim A'_\nu) \cot \beta &= \sin \zeta'_s \cot \zeta'_\nu - \cos (A'_s \sim A'_\nu) \cos \zeta'_s \end{aligned}\right\} \qquad (47)$$

Considering all the parts as variable, differentiating and reducing, we find

$$\left.\begin{aligned} d\varrho &= \sin \zeta'_\nu \sin \gamma \, d(A'_s \sim A'_\nu) + \cos \gamma \, d\zeta'_\nu + \cos \beta \, d\zeta'_s \\ -\sin \varrho \, d\beta &= \sin \zeta'_\nu \cos \gamma \, d(A'_s \sim A'_\nu) - \sin \gamma \, d\zeta'_\nu + \cos \varrho \sin \beta \, d\zeta'_s \end{aligned}\right\} \qquad (48)$$

Reverting to the equations (26), putting $\pi'_s = M\pi_s$, $\pi'_v = N\pi_v$, $\pi_s = m\pi_v$, where π_s and π_v are respectively the equatorial horizontal parallaxes of the sun and Venus; regarding π_v as the variable, and differentiating, we get

$$\left.\begin{aligned} d\zeta'_s &= +\ mM\cos\theta_s\, d\pi_v \\ dA'_s &= \pm\ mM\sin\theta_s\, d\pi_v \\ d\zeta'_v &= +\ N\cos\theta_v\, d\pi_v \\ dA'_v &= \pm\ N\sin\theta_v\, d\pi_v \end{aligned}\right\} \tag{49}$$

Substituting these values in (48), writing unity for sin ζ'_v, θ_s for θ_v, and $d\omega$ for $-\,d\beta$, we obtain

$$\left.\begin{aligned} d\varrho &= \cos\theta_s\,[\pm(mM\pm N)\sin\gamma\tan\theta_s + N\cos\gamma + mM\cos\beta]\,d\pi_v \\ d\omega &= \frac{\cos\theta_s}{\sin\rho}[\pm(mM\pm N)\cos\gamma\tan\theta_s - N\sin\gamma + mM\sin\beta\cos\varrho]\,d\pi_v \end{aligned}\right\} \tag{50}$$

From an ephemeris the polar distances of the sun and Venus, and the difference of their right ascensions are taken; and thus two sides and the included angle are known in the triangle PSV. The remaining parts are given by the formulæ

$$\left.\begin{aligned} \tan\mu &= \tan\Delta_v\cos(\alpha_s\sim\alpha_v) \\ \sin\varrho_0\cos\omega_0 &= \frac{\cos\Delta_v}{\cos\mu}\sin(\Delta_s-\mu) \\ \sin\varrho_0\sin\omega_0 &= \sin\Delta_v\sin(\alpha_s\sim\alpha_v) \\ \sin\varrho_0\sin\sigma &= \sin\Delta_s\sin(\alpha_s\sim\alpha_v) \end{aligned}\right\} \tag{51}$$

in which we write ϱ_0 and ω_0 to distinguish the quantities deduced from the ephemeris from the similar quantities ϱ and ω obtained from the photographs by means of the equations (30), (32), and (34).

Still considering the triangle PSV, we have the relations

$$\left.\begin{aligned} \cos\varrho_0 &= \cos\Delta_v\cos\Delta_s + \sin\Delta_v\sin\Delta_s\cos(\alpha_s\sim\alpha_v) \\ \sin(\alpha_s\sim\alpha_v)\cot\omega_0 &= \sin\Delta_s\cot\Delta_v - \cos\Delta_s\cos(\alpha_s\sim\alpha_v) \end{aligned}\right\} \tag{52}$$

Regarding all the parts as variable, differentiating and reducing, we get

$$\left.\begin{aligned} d\varrho_0 &= \sin\Delta_v\sin\sigma d(\alpha_s\sim\alpha_v) + \cos\sigma d\Delta_v + \cos\omega_0 d\Delta_s \\ -\sin\varrho_0 d\omega_0 &= \sin\Delta_v\cos\sigma d(\alpha_s\sim\alpha_v) - \sin\sigma d\Delta_v + \cos\varrho_0\sin\omega_0 d\Delta_s \end{aligned}\right\} \tag{53}$$

If now α_s, α_v, Δ_s, Δ_v, π_s, and π_v are the tabular values of the right ascensions, polar distances, and parallaxes of the sun and Venus; and if $\alpha_s + d\alpha_s$, $\alpha_v + d\alpha_v$, $\Delta_s + d\Delta_s$, $\Delta_v + d\Delta_v$, $\pi_s + d\pi_s$, $\pi_v + d\pi_v$, are the true values of the same quantities; then we must have

$$\left.\begin{aligned} \varrho + d\varrho &= \varrho_0 + d\varrho_0 \\ \omega + d\omega &= \omega_0 + d\omega_0 \end{aligned}\right\} \qquad (54)$$

Substituting the values of $d\varrho$, $d\varrho_0$, $d\omega$, and $d\omega_0$ from (50) and (53), we obtain finally

$$0 = \left\{\begin{aligned} &\rho_0 - \rho + \sin \Delta_v \sin \sigma d(\alpha_s \sim \alpha_v) + \cos \sigma d\Delta_v + \cos \omega_0 d\Delta_s \\ &- \cos \theta_s [\pm (mM \pm N) \sin \gamma \tan \theta_s + N \cos \gamma + mM \cos \beta] d\pi_v \end{aligned}\right\} \quad (55)$$

$$0 = \left\{\begin{aligned} &\omega_0 - \omega - \frac{1}{\sin \rho_0} [\sin \Delta_v \cos \sigma d(\alpha_s \sim \alpha_v) - \sin \sigma d\Delta_v + \cos \rho_0 \sin \omega_0 d\Delta_s] \\ &- \frac{\cos \theta_s}{\sin \rho} [\pm (mM \pm N) \cos \gamma \tan \theta_s - N \sin \gamma + mM \sin \beta \cos \rho] d\pi_v \end{aligned}\right\} \quad (56)$$

Each photograph furnishes one conditional equation of the form (55), and another of the form (56), and from all the equations thus obtained the values of $d(\alpha_s \sim \alpha_v)$, $d\Delta_s$, $d\Delta_v$, and $d\pi_v$ are found by the method of least squares, the resulting value of the solar parallax being

$$\pi_s + m \, d\pi_v \qquad (57)$$

At the time of the last transit the value of m was 0.2684. The term $\pm (mM \pm N)$ of equations (55) and (56) is to be interpreted thus: When the sun and Venus are on opposite sides of the meridian, it will be $+ (mM + N)$; when the sun and Venus are on the same side of the meridian, if the sun is most distant from the meridian, it will be $+ (mM - N)$; but if Venus is most distant from the meridian, then it will be $- (mM - N)$.

It will not escape notice that those parts of equations (55) and (56) which correspond to $\varrho_0 + d\varrho_0$, and $\omega_0 + d\omega_0$, of the equations (54), are general for the whole earth, and can therefore be tabulated at suitable intervals for the period of the transit; while the terms which correspond to $\varrho + d\varrho$, and $\omega + d\omega$, must be computed specially for each photograph.

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